# Interaction of Cosmic Strings with Gravitational Waves: A New Class of Exact Solutions

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A new three-parameter family of radiative, cylindrically symmetric solutions of the Einstein vacuum equations is presented. Its members represent analytic models of a cylindrical pulse of gravitational radiation reflecting off a straight-line cosmic string.

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Among the nontrivial topological structures that grand unified theories with spontaneous symmetry breaking predict to have formed in the very early Universe, cosmic strings are supposed to be the ones with the highest probability of surviving up to the present cosmological era.[1] This conclusion has led to an extensive effort to find characteristic effects which would provide the basis for verification of the existence of cosmic strings observationally.

The first important result in this direction was obtained by Vilenkin.[2] On the basis of an approximate solution of the Einstein field equations, he concluded that a straight-line cosmic string of vanishing thickness will give rise to the effect of lensing or multiple image production, even though no gravitational field appears in the string's vicinity. This is due to the fact that, in the presence of a cosmic string of the above type, the space-time manifold remains flat but assumes a conical structure whereby its angular spread around the string becomes less than $2\pi$ rad.

Subsequent studies[3] based on exact solutions of the Einstein equations corresponding to finite cross-section cylinders as models of open cosmic strings have given ample support to Vilenkin's main result. Only the relation $D=8\pi\mu$ between the angular defect $D$ of the space-time manifold in which a straight-line cosmic string is embedded and the latter's linear mass density $\mu$ has been questioned so far.[4] Most of the above studies, however, have been based on static models of the string's space-time background. But this background is subject to the disturbances produced by gravitational waves emitted by other objects in the string's vicinity. On the other hand, the "false-vacuum" region represented by a cosmic string must not be expected to be homogeneous from the very beginning of the string's formation. Thus, a relaxation process can be envisaged whereby, as the string evolves to its "lower-energy configuration" (probably, represented by one of the static models mentioned above) gravitational waves are emitted.[5] Therefore, the development of a more realistic picture of the gravitational effects associated with cosmic strings requires the construction of the appropriate time-dependent models.

In this Letter we present an analytic model of a Vilenkin-type cosmic string interacting with cylindrical gravitational waves carrying both of the degrees of freedom that are possible for these kinds of waves. More specifically, the model represents a pulse of gravitational radiation reflecting off a straight-line cosmic string which occupies the axis of a cylindrically symmetric space-time. Its construction is based on a new class of exact radiative solutions of the Einstein vacuum equations which depends on three parameters ($\alpha$, $\beta$, and $\gamma$, below), two of which are free. Analytic models of straight-line cosmic strings interacting with gravitational waves have also been presented by Xanthopoulos,[6] Garriga and Verdaguer,[7] and Economou and Tsoubelis.[8] What distinguishes the model presented below from the ones just mentioned is a set of features of which the following are worth pointing out. As noted above, the gravitational waves involved in the present model have both of the degrees of freedom which cylindrical waves can carry. These degrees of freedom are referred to as the + and $x$ polarization modes, respectively, and when both are present it is not possible to find a coordinate system in which the space-time metric is globally diagonal. As a result, the corresponding Einstein equations are so hard to integrate that no analytic solution of this type was known as late as in 1985, when Piran, Safier, and Stark[9] published a thorough numerical analysis of the behavior of such solutions. Things are much simpler when only the + mode is present. In this case the metric can be rendered diagonal [let $X=0$ in Eq. (5), below] and the corresponding solutions are known as Einstein-Rosen[10] waves. It is these kinds of waves that appear in the Garriga-Verdaguer models mentioned above.

Within the category of the nondiagonal solutions, on the other hand, the models presented below have the advantage of being the first ones to have the following, physically important, characteristics. First, the global parameter $\alpha$ which measures the strength of the cosmic string is uncoupled from the parameters expressing the characteristics of the gravitational wave. Second, the parameter $\beta$ which represents the strength of the gravitational wave is freely variable within a range which includes the value $\beta=0$ at which the wave is switched off. One of the consequences of the above characteristics is to render the physical interpretation of the present solution unambiguous and, thereby, to answer some questions regarding the physical meaning of similar solutions obtained by Xanthopoulos[6] and the present authors[8] recently in the manner suggested in Ref. 8.

The new class of solutions described above can be derived most conveniently by the application of a limiting procedure to the line element

$$ds^2=c^2X[-x^2+1)^{-1}dx^2+(y^2-1)^{-1}dy^2]+(X/Y)(x^2+1)(y^2-1)d\phi^2+(Y/X)(dz-\omega d\phi)^2, \quad (1)$$

where

$$X=(1-px)^2+(l-qy)^2, \quad Y=p^2(x^2+1)+q^2(y^2-1),$$

$$\omega=(2/pY)[lp^2(y-1)(x^2+1) +q(1+l^2-lq-px)(y^2-1)]. \quad (2)$$

The range of the coordinates appearing in Eq. (4) is given by $x\in\mathbb{R}$, $y\in[1,\infty)$, $\phi\in[0,2\pi)$, and $z\in\mathbb{R}$, while $c$, $p$, $l$, and $q$ are real parameters, the last three of which satisfy the condition

$$q^2-l^2-p^2=1. \quad (3)$$

As shown in Ref. 8, the above line element represents a cylindrically symmetric, Petrov type-D solution of the Einstein vacuum equations. The cylindrical symmetry of the metric can be made explicit by the coordinate transformation

$$t=xy, \quad \rho=[(x^2+1)(y^2-1)]^{1/2}, \quad (4)$$

which brings the line element given by Eqs. (1) and (2) into the Jordan-Ehlers-Kompaneets canonical form[11]

$$ds^2=f(-dt^2+d\rho^2)+e^{-2\Psi}\rho^2d\phi^2 +e^{2\Psi}(dz-\chi d\phi)^2, \quad (5)$$

where

$$f=c^2(x^2+y^2)^{-1}X, \quad e^{2\Psi}=Y/X, \quad \chi=\omega, \quad (6)$$

and $x,y$ are expressed in terms of $t$ and $\rho$ according to Eq. (4). When $f=1$ and $\Psi=\omega=0$, Eq. (5) gives the line element of Minkowski space-time in cylindrical coordinates. This makes clear the meaning of the various coordinate systems that appear in our analysis.

Now, let $(\alpha,\beta,\gamma)$ be the set of parameters defined by the equation

$$(c,l,p)=(\alpha/\gamma q,\beta q,\gamma q), \quad (7)$$

and consider the limit $q\to\infty$ of Eqs. (3) and (6). If Eq. (2) is taken into account, it is easy to verify that the resulting expressions read

$$\beta^2+\gamma^2=1, \quad (8)$$

and

$$f=(\alpha/\gamma)^2(x^2+y^2)^{-1}A, \quad e^{2\Psi}=B/A,$$

$$X=(2\beta/\gamma B)[\gamma^2(y-1)(x^2+1)-(1-\beta)(y^2-1)], \quad (9)$$

$$A\equiv\gamma^2x^2+(y-\beta)^2, \quad B\equiv\gamma^2(x^2+1)+(y^2-1),$$

respectively. Equations (4), (5), (8), and (9) give the new three-parameter class of solutions announced above. The corresponding metric retains the Petrov type-D character of the one we started with and, provided $|\beta|\neq1$, it is regular everywhere except on the symmetry axis where, in general, it is quasiregular.[12] In equivalent terms, the axis region of a typical member of the new class of space-time models constructed above has a conical structure which reveals itself in the angular defect to be calculated shortly. On the basis of the results quoted in the introduction, we attribute this behavior to the presence of a straight-line cosmic string occupying the symmetry axis.

In order to unveil the physical meaning of the new solution let us first set $\beta=0$ to obtain

$$ds^2=\alpha^2(-dt^2+d\rho^2)+\rho^2d\phi^2+dz^2. \quad (10)$$

Thus, by choosing $\beta=0$ and $|\alpha|>1$ we end up with a flat space-time characterized by an angular defect $D_<$, where

$$D_<=2\pi(1-|\alpha|^{-1}). \quad (11)$$

Next, let $\beta\neq0$ and consider the region near the symmetry axis $\rho=0$. In this region $\rho\ll|t|$. Thus,

$$x\approx t-\rho^2t/2(t^2+1), \quad y\approx1+\rho^2t/2(t^2+1), \quad (12)$$

which, together with Eq. (9), implies that

$$ds^2\approx\alpha^2F[-dt^2+d\rho^2+(\rho/\alpha)^2d\phi^2]+F^{-1}dz^2, \quad (13)$$

where

$$F=[(\beta-1)^2+\gamma^2t^2]/\gamma^2(t^2+1). \quad (14)$$

Therefore, for $|t|\gg1$, i.e., for very early and late times, the metric in the axis region approaches the one given by Eq. (10).

Let us, now, turn our attention to the asymptotic region $\rho\gg1,|t|$. Here,

$$x\approx t/\rho+t(t^2-1)/2\rho^3, \quad y\approx\rho+(1-t^2)/2p, \quad (15)$$

which implies that

$$ds^2\approx(\alpha/\gamma)^2[-dt^2+d\rho^2+(\gamma/\alpha)^2\rho^2d\phi^2] +[dz-2(\beta/\gamma)(\beta-1)d\phi]^2 \quad (16)$$

Therefore, the angular defect measured at spacelike infinity is given by

$$D_>=2\pi(1-|\gamma/\alpha|). \quad (17)$$

This is larger than the angular defect measured near the axis by an amount $\delta D$, where

$$\delta D \equiv D_> - D_< = 2\pi(1-|\gamma|)/|\alpha| > 0\,, \qquad (18)$$

unless $\beta=0$, in which case $|\gamma|=1$ and $\delta D$ vanishes. We will now show that the enhancement of the angular defect expressed by Eq. (18) is due to a pulse of cylindrical waves which, at $t=-\infty$, approaches the axis along the null direction $t=-\rho$, gets reflected off the axis region at $t=0$, and, eventually, recedes to the asymptotic region $\rho=\infty$ along the outgoing null direction $t=\rho$.

In order to prove the above claim, let us consider the quantity

$$C(t,\rho) \equiv \tfrac{1}{2}\ln(fe^{2\Psi})\,, \qquad (19)$$

which is referred to as Thorne's $C$ energy.[13] For cylindrically symmetric systems, it provides a well defined measure of the energy per unit length in the $z$ direction confined within a region of coordinate radius $\rho$. From Eq. (9), it follows that in our case

$$2C = \ln(\alpha/\gamma)^2 \left[1 - \beta^2 \frac{x^2+1}{x^2+y^2}\right]. \qquad (20)$$

Introducing the null coordinates $u$ and $v$, where

$$u = t-\rho, \quad v = t+\rho\,, \qquad (21)$$

we find that

$$2C = \ln(\alpha/\gamma)^2 \left[1 - \frac{\beta^2}{2}\left(1 + \frac{1+uv}{E}\right)\right],$$
$$E \equiv [(1+u^2)(1+v^2)]^{1/2}\,, \qquad (22)$$

and

$$2\left(\frac{\partial C}{\partial u}, \frac{\partial C}{\partial v}\right) = \frac{\beta^2(u-v)}{(2-\beta^2)E - \beta^2(1+uv)]}\left(\frac{1}{1+u^2}, \frac{-1}{1+v^2}\right). \qquad (23)$$

Equations (22) and (23) imply that on the symmetry axis, where $u=v$, the $C$ energy is constant and equal to $C_<$, where

$$C_< = \ln|\alpha|\,. \qquad (24)$$

Comparing this with Eq. (11) we conclude that a nonvanishing angular defect near the axis and, therefore, the presence of a cosmic string, is equivalent to the $C$ energy in this region being positive definite.

Let it, now, be noted that the $C$ energy is constant and equal to $C_<$ in the regions $\mathrm{I}^-$, where $v \ll -1$, and $\mathrm{I}^+$, where $u \gg 1$, as well. Similarly the $C$ energy is constant and equal to $C_>$, where

$$C_> = \ln|\alpha/\gamma|\,, \qquad (25)$$

in the asymptotic region III, where $u \ll -1$, $v \gg 1$. But this ceases to be the case in region II which has the symmetry axis and regions I and III as its boundaries. In fact, in subregion $\mathrm{II}^-$ where $u<0$, i.e., in a world tube centered on the radial null direction $v=0$, there is an energy flux towards the axis, while in $\mathrm{II}^+$ where $v>0$ there is an energy flux away from the axis of symmetry. In order to determine the shape of the ingoing pulse, consider a null direction along which $v=$const and use Eq. (23) to find that

$$\lim_{u\to-\infty}\left(\frac{\partial C}{\partial u}\right)=0, \quad \lim_{u\to-\infty}\left(\frac{\partial C}{\partial v}\right) = \frac{\beta^2}{2(1+v^2)[(2-\beta^2)(1+v^2)^{1/2}+\beta^2 v]}. \qquad (26)$$

Equation (26) shows clearly that at $t=-\infty$ there is indeed a flux of gravitational radiation towards $\rho=0$ which is concentrated around the null direction $t=-\rho$. Similarly, along $u=$const

$$\lim_{v\to\infty}\left(\frac{\partial C}{\partial u}\right) = \frac{-\beta^2}{2(1+u^2)[(2-\beta^2)(1+u^2)^{1/2}-\beta^2 u]}, \quad \lim_{v\to\infty}\left(\frac{\partial C}{\partial u}\right)=0\,. \qquad (27)$$

Thus, an outgoing pulse appears as $t\to\infty$ which is identical in form with the ingoing one at $t=-\infty$.

The above analysis shows clearly that an angular defect appears in the wake of the pulse of gravitational waves incident on the string occupying the axis. As a result, after the pulse has crossed the cylindrical surface $\rho=\rho_0 \gg 1$ inwards, the angular defect measured at $\rho>\rho_0$ is found to have increased by $\delta D$. This increase which, according to Eqs. (18), (24), and (25), is given by

$$\delta D = 2\pi[\exp(-C_<) - \exp(-C_>)]\,, \qquad (28)$$

lasts for as long as the pulse is confined in the region $\rho<\rho_0$, i.e., for $\delta t \approx 2\rho_0$. Obviously this process is in accord with the intuitively expected result that the angular defect produced by a dynamic, cylindrically symmetric system will increase (decrease) as a result of the system's absorption (emission) of gravitational radiation.

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